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ASYMPTOTIC EXPANSIONS FOR A CLASS OF HYPERGEOMETRIC FUNCTIONS

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Saba Mudaliar

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The usual power series representations for hypergeometric functions in two variables have a limited range of validity. In particular, they are of little use when the magnitude of one of the variables becomes very large. Using a Barnes-type integral representation and the concept of analytic continuation, the region of utility is extended to the desired domain. The poles that occur in the Barnes-type integral are assumed to be simple. Thus, explicit asymptotic expansions are obtained for each of the fourteen hypergeometric functions that belong to this class.

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Justification

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Dist	Avail and / or Special
A-1	

Illustrations

ASYMPTOTIC EXPANSIONS FOR A CLASS OF HYPERGEOMETRIC FUNCTIONS

1. INTRODUCTION

The study of hypergeometric functions is an old one and may be traced back to Gauss. Since then this function has been extensively analyzed and a comprehensive account of all the major results is given by Erdelyi et al [1953][1]. This subject is so rich and colorful that decades of research do not seem to have exhausted interest in it - investigators continue to find numerous applications in various disciplines. It is pleasing to note that most elementary and special functions can be expressed in terms of hypergeometric functions. What is more fascinating is that several obscure integrals have simple representations in terms of hypergeometric functions. Thus,it is still of interest to researchers to explore further the properties of and inter-relations between various hypergeometric functions.

Appell [Appell and Kampé de Fériet, 1926][2] extended the notion of hypergeometric function in a single variable to two variables. He defined four types of such functions. Later, Horn [1931][3] discovered a few more and a complete list of all the hypergeometric functions in two variables is given by Erdelyi et al [1953][1]. There are some errors in

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this list and the corrected versions are given in Srivastava and Karlsson [1985][4]

In this report we are concerned with the asymptotic series expansions of these hypergeometric functions when the magnitude of one of the variables becomes large. The basic concept is that of analytic continuation using a Barnes-type integral representation.

2. ANALYSIS

We begin by examining a Barnes-type integral suitable for the hypergeometric function under investigation. Depending on the contour of integration we obtain two types of expansions; this leads to the desired representation of the hypergeometric function. Detailed analysis of this procedure is provided for the first hypergeometric function $F_1(\alpha,\beta,\tilde{\beta};\gamma;x,y)$. For the other thirteen hypergeometric functions that belong to this class the discussion is brief since the procedures are similar.

Consider

$$I_+ = \frac{1}{2\pi i}\int_{C_+} \frac{\Gamma(\alpha+m+z)}{\Gamma(\gamma+m+z)}\,\Gamma(\tilde{\beta}+z)\,\Gamma(-z)\,(-y)^z\,dz \qquad (1)$$

where C_+ is the semicircular contour on the right hand side (see Figure 1) of the z-plane; $\alpha,\tilde{\beta},\gamma$, and y are complex while m is an integer. As shown in the figure the contour is indented in such a way that the sequence of poles extending to the right ($z = n$; $n = 0, 1, 2, \ldots$) is separated from that extending to the left ($z = -n-\tilde{\beta}$;

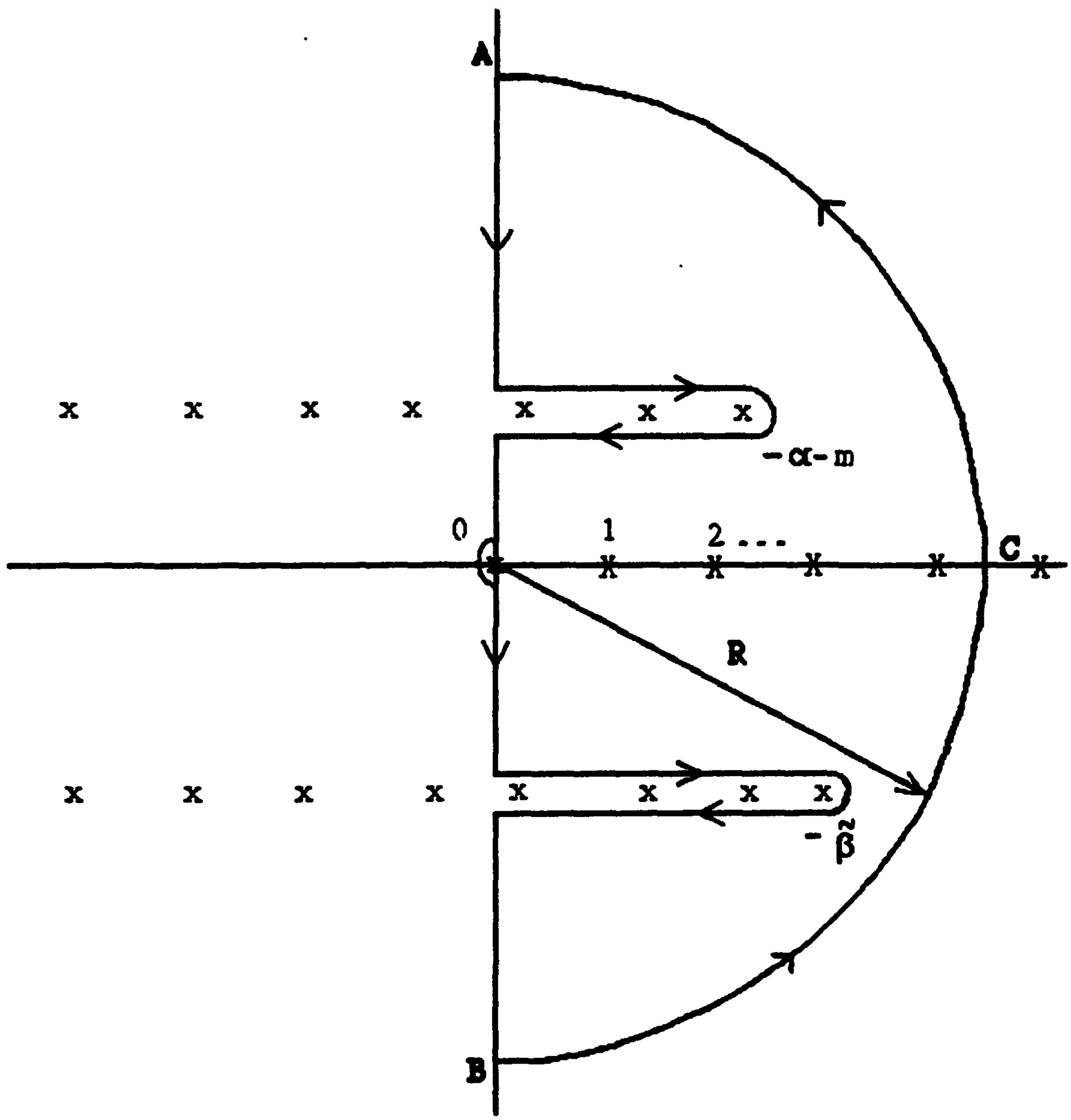

Figure 1. Contour of Integration C_+

$z = -n-\alpha-m$ for $n = 0, 1, 2, \ldots$). Let the radius of the semicircle be R. Also we assume that all the poles are simple.

$$2\pi i\, I_+ = \int_{AB} + \int_{BCA} = -I + J_+ \tag{2}$$

Consider first

$$J_+ = \int_{-\pi/2}^{\pi/2} \frac{\Gamma(\alpha+m+z)}{\Gamma(\gamma+m+z)} \Gamma(\tilde{\beta}+z)\, \Gamma(-z)\, (-y)^z\, d(Re^{i\theta}) \tag{3}$$

On using the identity

$$\Gamma(1+z)\, \Gamma(-z) = -\pi\, \mathrm{cosec}(\pi z) \tag{4}$$

$$J_+ = \int_{-\pi/2}^{\pi/2} iRe^{i\theta}(-\pi)\, \frac{\Gamma(\alpha+m+z)\, \Gamma(\tilde{\beta}+z)}{\Gamma(\gamma+m+z)\, \Gamma(1+z)}\, \mathrm{cosec}(\pi z)\, (-y)^z\, d\theta$$

$$\leq \int_{-\pi/2}^{\pi/2} \pi R \left| \frac{\Gamma(\alpha+m+z)\, \Gamma(\tilde{\beta}+z)}{\Gamma(\gamma+m+z)\, \Gamma(1+z)} \right| \left|\mathrm{cosec}(\pi z)\right| \left|(-y)^z\right| d\theta \tag{5}$$

Now the asymptotic form of the gamma function is given as [Abramovitz and Stegun, 1964][5]

$$\Gamma(az+b) \sim \sqrt{(2\pi)}\, e^{-az}\, (az)^{az+b-.5} \quad , |z| \to \infty \ , |\arg(z)| < \pi \ , a > 0 \tag{6}$$

Thus

$$\left| \frac{\Gamma(\alpha+m+z)\ \Gamma(\tilde{\beta}+z)}{\Gamma(\gamma+m+z)\ \Gamma(1+z)} \right| \sim R^{\alpha'+\tilde{\beta}'-\gamma'-1} \quad \exp\left(-\theta(\alpha''+\tilde{\beta}''-\gamma'')\right) \tag{7}$$

where $'$ and $''$ denote respectively the real and imaginary parts of a complex quantity. Also

$$\left|\operatorname{cosec}(\pi z)\right| \sim 2\ e^{-\pi|z''|} \tag{8}$$

and

$$\left|(-y)^{z}\right| \sim |y|^{z'}\ e^{-\phi z''} \tag{9}$$

where

$$\phi = \arg(-y) \tag{10}$$

Thus the integrand of Eq. (5) (call it T_1) becomes

$$T_1 \sim \quad A_1\ R^{\alpha'+\tilde{\beta}'-\gamma'}\ |y|^{R\cos\theta}\ \exp\left\{-R|\sin\theta|\ (\pi+\phi)\right\} \qquad 0 < \theta < \pi/2$$

$$\sim \quad A_1\ R^{\alpha'+\tilde{\beta}'-\gamma'}\ |y|^{R\cos\theta}\ \exp\left\{-R|\sin\theta|\ (\pi-\phi)\right\} \qquad 0 > \theta > -\pi/2 \tag{11}$$

where A_1 is independent of R.

Now as $R \to \infty$

$$T_1 \to 0 \qquad \text{if } |y| < 1 \text{ and } \quad |\phi| < \pi \tag{12}$$

Also as $R \to \infty$

$$I = \int_{-i\infty}^{i\infty} dz \, \frac{\Gamma(\alpha+m+z)}{\Gamma(\gamma+m+z)} \, \Gamma(\tilde{\beta}+z) \, \Gamma(-z) \, (-y)^z \tag{13}$$

Evaluating Eq. (1) by the Cauchy integral theorem

$$I_+ = \sum_{n=0}^{\infty} \text{Residue at } z = n$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+m+n)}{\Gamma(\gamma+m+n)} \, \Gamma(\tilde{\beta}+n) \, \frac{(-1)^{n-1}}{n!} \, (-y)^n$$

$$= - \frac{\Gamma(\alpha) \, \Gamma(\tilde{\beta})}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} \, (\tilde{\beta})_n \, \frac{y^n}{n!} \tag{14}$$

where we use the Pochhamer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{15}$$

Thus from Eqs. (14), (13), (2), (12)

$$I = - 2\pi i \, \frac{\Gamma(\alpha) \, \Gamma(\tilde{\beta})}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} \, (\tilde{\beta})_n \, \frac{y^n}{n!} \tag{16}$$

Consider now

$$I_- = \frac{1}{2\pi i} \int_{C_-} \frac{\Gamma(\alpha+m+z)}{\Gamma(\gamma+m+z)} \Gamma(\tilde{\beta}+z)\ \Gamma(-z)\ (-y)^z\ dz \tag{17}$$

where C_ is the semicircular contour (AODB) of radius R in the left half of the z-plane (see Figure 2).

As before

$$2\pi i\ I_- = -I - J_- \tag{18}$$

where

$$J_- = \int_{\pi/2}^{3\pi/2} \frac{\Gamma(\alpha+m+z)}{\Gamma(\gamma+m+z)} \Gamma(\tilde{\beta}+z)\ \Gamma(-z)\ (-y)^z\ d(Re^{i\theta}) \tag{19}$$

Using the identity of Eq. (4)

$$J_- = \int_{\pi/2}^{3\pi/2} iRe^{i\theta} \frac{\Gamma(1-\gamma-m-z\ )}{\pi\ \mathrm{cosec}[\pi(\gamma+m+z)]} \cdot \left| \frac{\mathrm{cosec}[\pi(\alpha+m+z)]\ \mathrm{cosec}[\pi(\tilde{\beta}+z)]}{\mathrm{cosec}[\pi(\gamma+m+z)]} \right| \left|(-y)^z\right|\ d\theta$$

$$\le \int_{\pi/2}^{3\pi/2} R\pi \left| \frac{\Gamma(1-\gamma-m-z)\ \Gamma(-z)}{\Gamma(1-\alpha-m-z)\ \Gamma(1-\tilde{\beta}-z)} \right|$$

$$\cdot \frac{\pi\ \mathrm{cosec}[\pi(\beta+z)]}{\Gamma(1-\tilde{\beta}-z)} \frac{\pi\ \mathrm{cosec}[\pi(\alpha+m+z)]}{\Gamma(1-\alpha-m-z)} \Gamma(-z)\ (-y)^z\ d\theta$$

(20)

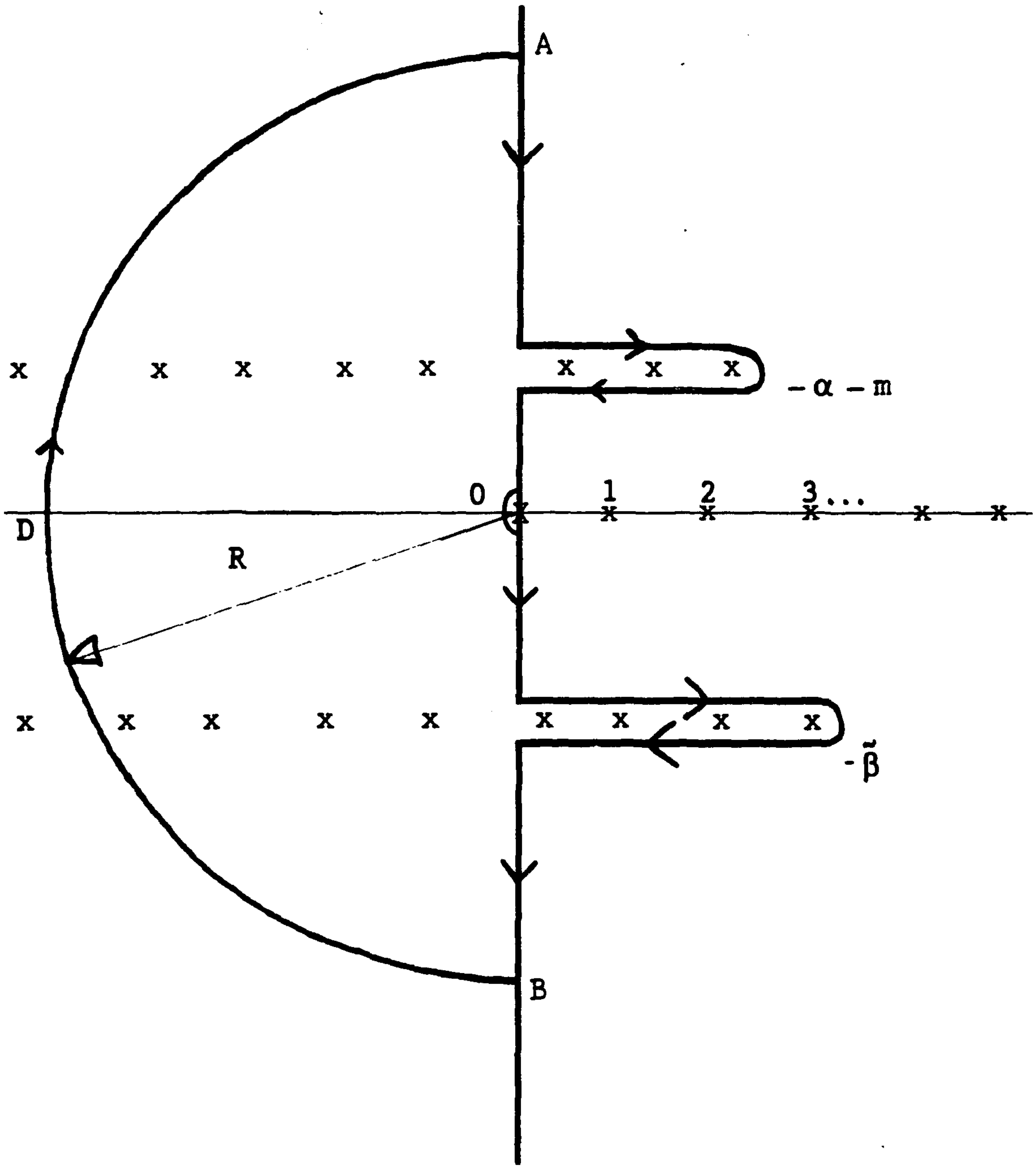

Figure 2. Contour of Integration C₋

and since

$$\left|\Gamma(a-z)\right| \sim \sqrt{(2\pi)}\ e^{-z'} R^{a'-z'}\ R^{-1/2} \exp[-\theta(a''-z'')]\ ,\quad \left|\arg(z)\right| > 0$$

(21)

$$\left|\frac{\Gamma(1-\gamma-m-z)\ \Gamma(-z)}{\Gamma(1-\alpha-m-z)\ \Gamma(1-\bar{\beta}-z)}\right| \sim R^{\alpha'+\bar{\beta}'-\gamma'-1} \exp[-\theta(\alpha''+\beta''-\gamma'')],\quad \left|\arg(z)\right| > 0$$

(22)

and

$$\left|\frac{\operatorname{cosec}[\pi(\alpha+m+z)]\ \operatorname{cosec}[\pi(\tilde{\beta}+z)]}{\operatorname{cosec}[\pi(\gamma+m+z)]}\right| \sim 2\ e^{-\pi|z''|}\ , \tag{23}$$

thus T_2. the integrand in Eq. (19) is given as

$$T_2 \sim A_2\ R^{\alpha'+\bar{\beta}'-\gamma'}\ |y|^{-R|\cos\theta|} \exp\left\{-R|\sin\theta|\ (\pi+\phi)\right\} \qquad \pi/2 < \theta < \pi$$

$$\sim A_2\ R^{\alpha'+\bar{\beta}'-\gamma'}\ |y|^{-R|\cos\theta|} \exp\left\{-R|\sin\theta|\ (\pi-\phi)\right\} \qquad \pi < \theta < 3\pi/2$$

(24)

where A_2 is independent of R.

It is clear then that as $R \to \infty$

$$T_2 \to 0 \quad \text{if } |y| > 1 \text{ and } |\arg(-y)| < \pi \tag{25}$$

Evaluating Eq. (19) by the Cauchy integral theorem

$$I_- = -\sum_{n=0}^{\infty} \text{Residue at } z = -n-\alpha-m$$

$$-\sum_{n=0}^{\infty} \text{Residue at } z = -n-\tilde{\beta} \tag{26}$$

From Eqs. (26), (25), (20) and (18)

$$I = \sum_{n=0}^{\infty} \text{Residue at } z = -n-\alpha-m$$

$$+\sum_{n=0}^{\infty} \text{Residue at } z = -n-\tilde{\beta} \qquad \text{if } |y| > 1 \text{ and } |\arg(-y)| < \pi \tag{27}$$

Thus for $|y| > 1$ we may analytically continue the RHS of Eq. (16) to the RHS of Eq. (27) as follows.

$$\frac{\Gamma(\alpha)\,\Gamma(\tilde{\beta})}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} (\tilde{\beta})_n \frac{y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \text{Residue at } z = -n-\alpha-m$$

$$+\sum_{n=0}^{\infty} \text{Residue at } z = -n-\tilde{\beta} \; ; \qquad |y| > 1 \text{ and } |\arg(-y)| < \pi \tag{28}$$

Residue at $z = -n-\alpha-m$

$$= -\frac{(-1)^n}{n!}\frac{\Gamma(\bar{\beta}-\alpha-m-n)}{\Gamma(\gamma-\alpha-n)}\Gamma(\alpha+m+n)\ (-y)^{-m-n-\alpha}$$

$$= -\frac{(-1)^{m+n}}{n!}\frac{(\alpha)_{m+n}\ (1+\alpha-\gamma)_n}{(1+\alpha-\bar{\beta})_{m+n}}\frac{\Gamma(\alpha)\ \Gamma(\bar{\beta}-\alpha)}{\Gamma(\gamma-\alpha)}(-y)^{-m-n-\alpha} \tag{29}$$

Residue at $z = -n-\bar{\beta}$

$$= -\frac{\Gamma(\alpha-\bar{\beta}+m-n)}{\Gamma(\gamma-\bar{\beta}+m-n)}\Gamma(\bar{\beta}+n)\frac{(-1)^n}{n!}(-y)^{-n-\bar{\beta}}$$

$$= -\frac{(\alpha-\bar{\beta})_{m-n}}{(\gamma-\bar{\beta})_{m-n}}(\bar{\beta})_n\ \frac{\Gamma(\bar{\beta})\ \Gamma(\alpha-\bar{\beta})}{\Gamma(\gamma-\bar{\beta})}\frac{(-1)^n}{n!}(-y)^{-n-\bar{\beta}}$$

(30)

Using Eqs. (29) and (30) in Eq. (28).

$$\frac{\Gamma(\alpha)\ \Gamma(\bar{\beta})}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_{m+n}}(\bar{\beta})_n\ \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\alpha)\ \Gamma(\bar{\beta}-\alpha)}{\Gamma(\gamma-\alpha)}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\ (1+\alpha-\gamma)_n}{(1+\alpha-\bar{\beta})_{m+n}}\frac{(-1)^{m+n}}{n!}\quad(-y)^{-m-n-\alpha}$$

$$-\frac{\Gamma(\bar{\beta})\ \Gamma(\alpha-\bar{\beta})}{\Gamma(\gamma-\bar{\beta})}\sum_{n=0}^{\infty}\frac{(\alpha-\bar{\beta})_{m-n}}{(\gamma-\bar{\beta})_{m-n}}(\bar{\beta})_n\ \frac{(-1)^n}{n!}(-y)^{-n-\bar{\beta}} \tag{31}$$

Multiplying both sides of Eq. (31) by $(\beta_m) \frac{x^m}{m!}$ and summing over $m = 0$ to ∞ we get

$$\frac{\Gamma(\alpha)\ \Gamma(\bar{\beta})}{\Gamma(\gamma)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} (\bar{\beta})_n\ (\beta)_m\ \frac{x^m}{m!}\ \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\alpha)\ \Gamma(\bar{\beta}-\alpha)}{\Gamma(\gamma-\alpha)} (-\ y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m\ \ (\alpha)_{m+n}\ \ (1+\alpha-\gamma)_n}{(1+\alpha-\bar{\beta})_{m+n}}\ \frac{(x/y)^m}{m!}\ \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(\bar{\beta})\ \Gamma(\alpha-\bar{\beta})}{\Gamma(\gamma-\bar{\beta})} (-\ y)^{-\bar{\beta}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta)_m \frac{(\alpha-\bar{\beta})_{m-n}}{(\gamma-\bar{\beta})_{m-n}} (\bar{\beta})_n\ \frac{x^m}{m!}\ \frac{(1/y)^n}{n!} \tag{32}$$

It may be noted that in our derivation m has been implicitly assumed to be a finite integer. But since the series in Eq.(32) is uniformly convergent we may let m become arbitrarily large.

The formal definition of the hypergeometric function $F_1(\alpha,\beta,\bar{\beta};\gamma:x,y)$ is given as [Erdelyi et al. 1953][1]

$F_1(\alpha,\beta,\tilde{\beta};\gamma;x,y)$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m \, (\alpha)_{m+n} \, (\tilde{\beta})_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{33}$$

Thus, we have the desired expansion for $F_1(\alpha,\beta,\tilde{\beta};\gamma;x,y)$ when $|x| < 1$ and $|y| > 1$

$F_1(\alpha,\beta,\tilde{\beta};\gamma;x,y)$

$$= - \frac{\Gamma(\tilde{\beta}-\alpha)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\tilde{\beta})} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m \, (\alpha)_{m+n} \, (1+\alpha-\gamma)_n}{(1+\alpha-\tilde{\beta})_{m+n}} \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\alpha-\tilde{\beta})\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\tilde{\beta})} (-y)^{-\tilde{\beta}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta)_m \frac{(\alpha-\tilde{\beta})_{m-n}}{(\gamma-\tilde{\beta})_{m-n}} (\tilde{\beta})_n \frac{x^m}{m!} \frac{(1/y)^n}{n!} \tag{34}$$

where $|\arg(-y)| < \pi$. The above procedure may be adopted to obtain asymptotic expansions for other hypergeometric functions belonging to this class. To obtain the expansion for $F_2(\alpha,\beta,\tilde{\beta};\gamma,\tilde{\gamma};x,y)$ we consider

$$I_+ = \frac{1}{2\pi i} \int_{C_+} \frac{\Gamma(\alpha+m+z)}{\Gamma(\tilde{\gamma}+z)} \Gamma(\tilde{\beta}+z)\,\Gamma(-z)\,(-y)^z \, dz \tag{35}$$

where C_+ is as before a semicircular contour to the right of the z-plane. On evaluating Eq. (35) we obtain for $|y| < 1$ and $|\arg(-y)\cdot < \pi$,

$$I = \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+m+z)}{\Gamma(\bar{\gamma}+z)} \Gamma(\bar{\beta}+z)\, \Gamma(-z)\, (-y)^z\, dz$$

$$= \frac{\Gamma(\alpha)\, \Gamma(\bar{\beta})}{\Gamma(\bar{\gamma})} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\, (\bar{\beta})_n}{(\bar{\gamma})_n}\, \frac{(-1)^n}{n!}\, (-y)^n \tag{36}$$

Next consider I_- which is the same integral as in Eq. (35) but along the contour C_-. This leads to

$$I = -\frac{\Gamma(\alpha)\, \Gamma(\bar{\beta}-\alpha)}{\Gamma(\bar{\gamma}-\alpha)} (-y)^{-\alpha}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\, (1+\alpha-\bar{\gamma})_{m+n}}{(1+\alpha-\bar{\beta})_{m+n}} (-1)^m\, y^{-m}\, \frac{y^{-n}}{n!}$$

$$- \frac{\Gamma(\bar{\beta})\, \Gamma(\alpha-\bar{\beta})}{\Gamma(\bar{\gamma}-\bar{\beta})} (-y)^{-\bar{\beta}}$$

$$\cdot \sum_{n=0}^{\infty} (\alpha-\bar{\beta})_{m-n}\, (1+\bar{\beta}-\bar{\gamma})_n\, (\bar{\beta})_n\, (-1)^n\, \frac{y^{-n}}{n!} \tag{37}$$

if $|y| > 1$ and $|\arg(-y)| < \pi$.

Thus, by analytic continuation

$$\frac{\Gamma(\alpha)\,\Gamma(\tilde{\beta})}{\Gamma(\tilde{\gamma})} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\tilde{\beta})_n}{(\tilde{\gamma})_n} \frac{(-1)^n}{n!} (-y)^n$$

$$= -\frac{\Gamma(\alpha)\,\Gamma(\tilde{\beta}-\alpha)}{\Gamma(\tilde{\gamma}-\alpha)} (-y)^{-\alpha}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(1+\alpha-\tilde{\gamma})_{m+n}}{(1+\alpha-\tilde{\beta})_{m+n}} (-1)^m\, y^{-m} \frac{y^{-n}}{n!}$$

$$- \frac{\Gamma(\tilde{\beta})\,\Gamma(\alpha-\tilde{\beta})}{\Gamma(\tilde{\gamma}-\tilde{\beta})} (-y)^{-\tilde{\beta}}$$

$$\cdot \sum_{n=0}^{\infty} (\alpha-\tilde{\beta})_{m-n}\,(1+\tilde{\beta}-\tilde{\gamma})_n\,(\tilde{\beta})_n\,(-1)^n \frac{y^{-n}}{n!} ;$$

$$|\arg(-y)| < \pi \qquad (38)$$

Multiplying Eq. (38) by $\frac{(\beta)_m}{(\gamma)_m} \frac{x^m}{m!}$ and summing the result over $m = 0$ to ∞ we get the desired asymptotic expansion.

$F_2(\alpha,\beta,\bar{\beta}:\gamma,\bar{\gamma}:x,y)$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_m\,(\bar{\beta})_n}{(\gamma)_m\,(\bar{\gamma})_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\bar{\beta}-\alpha)\,\Gamma(\bar{\gamma})}{\Gamma(\bar{\gamma}-\alpha)\,\Gamma(\bar{\beta})} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m\,(\alpha)_{m+n}\,(1+\alpha-\bar{\gamma})_{m+n}}{(\gamma)_m\,(1+\alpha-\bar{\beta})_{m+n}} (-1)^m \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(\alpha-\bar{\beta})\,\Gamma(\bar{\gamma})}{\Gamma(\bar{\gamma}-\bar{\beta})\,\Gamma(\alpha)} (-y)^{-\bar{\beta}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_m} (\alpha-\bar{\beta})_{m-n}\,(1+\bar{\beta}-\bar{\gamma})_n\,(\bar{\beta})_n\,(-1)^n \frac{x^m}{m!} \frac{(1/y)^n}{n!} ;$$

$$|y| > 1 \; ; \; |x| < 1 \; ; \; |\arg(-y)| < \pi \tag{39}$$

The conditions that appear in Eq. (39) occur in expansions of all the hypergeometric functions that we are going to consider. Hence, for the sake of brevity we shall hereafter omit these conditions.

Next consider

$$I = \sum_{m=0}^{\infty} (\alpha)_m (\beta)_m \frac{x^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\bar{\alpha}+z)}{\Gamma(\gamma+m+z)} \Gamma(\bar{\beta}+z) \Gamma(-z) (-y)^z \, dz \tag{40}$$

Closing the contour of integration by semicircles to the right and then to the left of the z-plane and evaluating, we obtain as before

$F_3(\alpha, \bar{\alpha}, \beta, \bar{\beta}; \gamma; x, y)$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\bar{\alpha})_n}{(\gamma)_{m+n}} (\beta)_m (\bar{\beta})_n \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\bar{\beta}-\bar{\alpha}) \Gamma(\gamma)}{\Gamma(\gamma-\bar{\alpha}) \Gamma(\bar{\beta})} (-y)^{-\bar{\alpha}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\bar{\alpha})_n}{(\gamma-\bar{\alpha})_{m-n} (1+\bar{\alpha}-\bar{\beta})_n} (-1)^n \frac{x^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\bar{\alpha}-\bar{\beta}) \Gamma(\gamma)}{\Gamma(\gamma-\bar{\beta}) \Gamma(\bar{\alpha})} (-y)^{-\bar{\beta}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\bar{\beta})_n}{(\gamma-\bar{\beta})_{m-n} (1+\bar{\beta}-\bar{\alpha})_n} (-1)^n \frac{x^m}{m!} \frac{(1/y)^n}{n!} \tag{41}$$

To obtain an asymptotic expansion for

$$F_4(\alpha,\beta;\gamma,\bar{\gamma};x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\bar{\gamma})_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{42}$$

we consider

$$I = \frac{\Gamma(\bar{\gamma})}{\Gamma(\alpha)\ \Gamma(\bar{\beta})} \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} \frac{x^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+m+z)}{\Gamma(\bar{\gamma}+z)} \Gamma(\beta+m+z)\ \Gamma(-z)\ (-y)^z\ dz \tag{43}$$

This leads to the desired expansion for F_4.

$F_4(\alpha,\beta;\gamma,\tilde{\gamma};x,y)$

$$= -\frac{\Gamma(\beta-\alpha)\ \Gamma(\bar{\gamma})}{\Gamma(\bar{\gamma}-\alpha)\ \Gamma(\beta)} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\ (1+\alpha-\bar{\gamma})_{m+n}}{(\gamma)_m\ (1+\alpha-\beta)_n} \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(\alpha-\beta)\ \Gamma(\bar{\gamma})}{\Gamma(\bar{\gamma}-\beta)\ \Gamma(\alpha)} (-y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{m+n}\ (1+\beta-\bar{\gamma})_{m+n}}{(\gamma)_m\ (1+\beta-\alpha)_n} \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

(44)

Next, consider

$$I = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} \sum_{m=0}^{\infty} (\bar{\beta})_m \frac{x^m}{m!} \Gamma(1-\bar{\beta}-m)$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+m+z)}{\Gamma(1-\bar{\beta}-m+z)} \Gamma(\beta-m+z)\,\Gamma(-z)\, y^z \, dz \tag{45}$$

This leads to

$G_1(\alpha,\beta,\bar{\beta}\,;x,y)$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m+n} (\beta)_{n-m} (\bar{\beta})_{m-n} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\beta-\alpha)\,\Gamma(1-\bar{\beta})}{\Gamma(1-\bar{\beta}-\alpha)\,\Gamma(\beta)} y^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\bar{\beta}+\alpha)_{2m+n}}{(1+\alpha-\beta)_{2m+n}} (-1)^{m+n} \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\beta+\alpha)\,\Gamma(1-\bar{\beta})}{\Gamma(1-\bar{\beta}-\beta)\,\Gamma(\alpha)} y^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha+\beta)_{2m-n} (\beta)_{n-m} (\beta+\bar{\beta})_n (-1)^m \frac{(xy)^m}{m!} \frac{(1/y)^n}{n!}$$

(46)

Consider next

$$I = \sum_{m=0}^{\infty} (\alpha)_m \frac{(-x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\bar{\alpha}+z)}{\Gamma(1-\bar{\beta}-m+z)} \Gamma(\beta-m+z)\ \Gamma(-z)\ y^z\ dz \tag{47}$$

This leads to

$G_2(\alpha, \bar{\alpha}, \beta, \bar{\beta}; x, y)$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_m\ (\bar{\alpha})_n\ (\beta)_{n-m} (\bar{\beta})_{m-n}\ \frac{x^m}{m!}\ \frac{y^n}{n!}$$

$$= -\frac{\Gamma(\beta-\bar{\alpha})\ \Gamma(1-\bar{\beta})}{\Gamma(1-\bar{\beta}-\bar{\alpha})\ \Gamma(\beta)}\ y^{-\bar{\alpha}}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\bar{\alpha}+\bar{\beta})_{m+n}}{(1+\bar{\alpha}-\beta)_{m+n}} (\alpha)_m\ (\bar{\alpha})_n\ (-1)^{m+n}\ \frac{x^m}{m!}\ \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(\bar{\alpha}-\beta)\ \Gamma(1-\bar{\beta})}{\Gamma(1-\bar{\beta}-\beta)\ \Gamma(\bar{\alpha})}\ y^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\bar{\alpha}-\beta)_{m-n}}{(1-\beta)_{m-n}} (\alpha)_m\ (\beta+\bar{\beta})_n\ (-1)^n\ \frac{(xy)^m}{m!}\ \frac{(1/y)^n}{n!} \tag{48}$$

Next we seek an expansion for

$$G_3(\alpha,\tilde{\alpha};x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{2n-m} (\tilde{\alpha})_{2m-n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{49}$$

To this end we consider

$$I = \sum_{m=0}^{\infty} \frac{(\tilde{\alpha})_{2m}}{(1-\alpha)_m} \frac{(-x)^m}{m!} \frac{\Gamma(1-\tilde{\alpha}-2m)}{\Gamma[.5(\alpha-m)]\,\Gamma[.5(\alpha-m)+.5]}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma[.5(\alpha-m)+z]}{\Gamma(1-\tilde{\alpha}-2m+z)} \Gamma[.5(\alpha-m)+.5+z]\,\Gamma(-z)\,(-y)^z\,dz \tag{50}$$

whence we obtain

$$G_3(\alpha,\tilde{\alpha};x,y)$$

$$= -(-y)^{-.5\alpha} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\tilde{\alpha})_{2m}\,(.5\alpha-.5m)_n}{(1-\alpha)_m}$$

$$\cdot \frac{\Gamma(1-\tilde{\alpha}-2m)\,\Gamma(.5-n)}{\Gamma[.5(\alpha-m+1)]\,\Gamma(1-\tilde{\alpha}-.5\alpha-1.5m-n)} \frac{(ix/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$-(-y)^{-.5(1+\alpha)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\tilde{\alpha})_{2m}\,(.5\alpha-.5m+.5)_n}{(1-\alpha)_m}$$

$$\cdot \frac{\Gamma(1-\tilde{\alpha}-2m)\,\Gamma(.5-n)}{\Gamma[.5(\alpha-m)]\,\Gamma(.5-\tilde{\alpha}-.5\alpha-1.5m-n)} \frac{(ix/y)^m}{m!} \frac{(1/y)^n}{n!} \tag{51}$$

$$H_1(\alpha,\beta,\gamma;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m-n}}{(\delta)_m} (\beta)_{m+n} (\gamma)_n \frac{x^m}{m!} \frac{y^n}{n!} \tag{52}$$

is our next function. So we consider

$$I = \sum_{m=0}^{\infty} \frac{1}{(\delta)_m} \frac{(-x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\gamma+z)}{\Gamma(1-\alpha-m+z)} \Gamma(\beta+m+z)\, \Gamma(-z)\, y^z\, dz \tag{53}$$

and we obtain

$$H_1(\alpha,\beta,\gamma;x,y)$$

$$= -\frac{\Gamma(1-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(1-\beta-\alpha)\,\Gamma(\gamma)}\, y^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{2m+n}\,(\beta)_{m+n}}{(1+\beta-\gamma)_{m+n}\,(\delta)_m} (-1)^n \frac{(x/y)^m}{m!} \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma)}{\Gamma(1-\alpha-\gamma)\,\Gamma(\beta)}\, y^{-\gamma}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha+\gamma)_{m+n}\,(\beta-\gamma)_{m-n} \frac{(\gamma)_n}{(\delta)_m} \frac{(x)^m}{m!} \frac{(1/y)^n}{n!}$$

(54)

Our next function is

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m-n}}{(\epsilon)_m} (\beta)_m (\gamma)_n (\delta)_n \frac{x^m}{m!} \frac{y^n}{n!} \tag{55}$$

On considering

$$I = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\epsilon)_m} \frac{(x)^m}{m!} \frac{\Gamma(1-\alpha-m)}{\Gamma(\gamma)\,\Gamma(\delta)}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\gamma+z)}{\Gamma(1-\alpha-m+z)} \Gamma(\delta+z)\,\Gamma(-z)\, y^z\, dz \tag{56}$$

we obtain

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)$$

$$= -\frac{\Gamma(1-\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(1-\alpha-\gamma)\,\Gamma(\delta)} y^{-\gamma}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha+\gamma)_{m+n} (\beta)_m (\gamma)_n}{(1+\gamma-\delta)_n (\epsilon)_m} (-1)^n \frac{(x)^m}{m!} \frac{(1/y)^n}{n!}$$

$$-\frac{\Gamma(1-\alpha)\,\Gamma(\gamma-\delta)}{\Gamma(1-\alpha-\delta)\,\Gamma(\gamma)} y^{-\delta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m (\alpha+\delta)_{m+n} (\delta)_n}{(\epsilon)_m (1+\delta-\gamma)_n} (-1)^n \frac{(x)^m}{m!} \frac{(1/y)^n}{n!} \tag{57}$$

The next function is

$$H_3(\alpha,\beta;\gamma;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} \ (\beta)_n}{(\gamma)_{m+n}} \ \frac{x^m}{m!} \ \frac{y^n}{n!} \tag{58}$$

On considering

$$I = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \ \Gamma(\beta)} \sum_{m=0}^{\infty} \frac{(x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+2m+z)}{\Gamma(\gamma+m+z)} \Gamma(\beta+z) \ \Gamma(-z) \ (-y)^z \ dz \tag{59}$$

we obtain the desired expansion

$$H_3(\alpha,\beta;\gamma;x,y)$$

$$= - \frac{\Gamma(\beta-\alpha) \ \Gamma(\gamma)}{\Gamma(\beta) \ \Gamma(\gamma-\alpha)} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} \ (1+\alpha-\gamma)_{m+n}}{(1+\alpha-\beta)_{2m+n}} (-1)^m \ \frac{(x/y^2)^m}{m!} \ \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\alpha-\beta) \ \Gamma(\gamma)}{\Gamma(\alpha) \ \Gamma(\gamma-\beta)} (-y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_n \ (\alpha-\beta)_{2m-n}}{(\gamma-\beta)_{m-n}} \ \frac{(x)^m}{m!} \ \frac{(1/y)^n}{n!} \tag{60}$$

The next function is

$$H_4(\alpha,\beta;\gamma,\delta;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} \, (\beta)_n}{(\gamma)_m \, (\delta)_n} \; \frac{x^m}{m!} \; \frac{y^n}{n!} \tag{61}$$

On considering

$$I = \frac{\Gamma(\delta)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}}{(\gamma)_m} \; \frac{1}{\Gamma(\alpha+2m)} \; \frac{(x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+2m+z)}{\Gamma(\delta+z)} \, \Gamma(\beta+z) \, \Gamma(-z) \, (-y)^z \, dz \tag{62}$$

we obtain

$$H_4(\alpha,\beta;\gamma,\delta;x,y)$$

$$= - \frac{\Gamma(\beta-\alpha) \, \Gamma(\delta)}{\Gamma(\beta) \, \Gamma(\delta-\alpha)} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} \, (1+\alpha-\delta)_{2m+n}}{(1+\alpha-\beta)_{2m+n} \, (\gamma)_m} \; \frac{(x/y^2)^m}{m!} \; \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\alpha-\beta) \, \Gamma(\delta)}{\Gamma(\alpha) \, \Gamma(\delta-\beta)} (-y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_n \, (\alpha-\beta)_{2m-n}}{(\gamma)_m} (1+\beta-\delta)_n (-1)^n \, \frac{(x)^m}{m!} \; \frac{(1/y)^n}{n!} \tag{63}$$

The next function is

$$H_5(\alpha,\beta;\gamma;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} (\beta)_{n-m}}{(\gamma)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{64}$$

On considering

$$I = \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}}{(1-\beta)_m} \frac{\Gamma(\gamma)}{\Gamma(\alpha+2m)\,\Gamma(\beta-m)} (-1)^m \frac{(x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+2m+z)}{\Gamma(\gamma+z)} \Gamma(\beta-m+z)\,\Gamma(-z)\,(-y)^z\,dz \tag{65}$$

we get

$$H_5(\alpha,\beta;\gamma;x,y)$$

$$= -\frac{\Gamma(\beta-\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)} (-y)^{-\alpha}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n} (1+\alpha-\gamma)_{2m+n}}{(1+\alpha-\beta)_{3m+n}} (-1)^m \frac{(x/y^2)^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(\alpha-\beta)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)} (-y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-\beta)_{3m-n}}{(1-\beta)_{m-n} (\gamma-\beta)_{m-n}} \frac{(xy)^m}{m!} \frac{(1/y)^n}{n!} \tag{66}$$

The next function is

$$H_6(\alpha,\beta,\gamma;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{2m-n} (\beta)_{n-m} (\gamma)_n \frac{x^m}{m!} \frac{y^n}{n!} \tag{67}$$

On considering

$$I = \frac{\Gamma(1-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{(x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta-m+z)}{\Gamma(1-\alpha-2m+z)} \Gamma(\gamma+z)\,\Gamma(-z)\,(y)^z\,dz \tag{68}$$

we obtain

$$H_6(\alpha,\beta,\gamma;x,y)$$

$$= -\frac{\Gamma(1-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(1-\alpha-\beta)\,\Gamma(\beta)} (y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{m+n} (\gamma-\beta)_{m-n}}{(1-\beta)_{m-n}} (-1)^n \frac{(xy)^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma)}{\Gamma(1-\alpha-\gamma)\,\Gamma(\beta)} (y)^{-\gamma}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha+\gamma)_{2m+n} (\gamma)_n}{(1+\gamma-\beta)_{m+n}} (-1)^{m+n} \frac{(x)^m}{m!} \frac{(1/y)^n}{n!} \tag{69}$$

And finally,

$$H_7(\alpha,\beta,\gamma;\delta;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m-n}}{(\delta)_m} (\beta)_n (\gamma)_n \frac{x^m}{m!} \frac{y^n}{n!} \tag{70}$$

On considering

$$I = \frac{\Gamma(1-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}}{(\delta)_m (1-\alpha)_{2m}} \frac{(x)^m}{m!}$$

$$\cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta+z)\,\Gamma(\gamma+z)}{\Gamma(1-\alpha-2m+z)} \Gamma(-z)\,(y)^z\,dz \tag{71}$$

we obtain

$$H_7(\alpha,\beta,\gamma;\delta;x,y)$$

$$= -\frac{\Gamma(1-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(1-\alpha-\beta)\,\Gamma(\gamma)} (y)^{-\beta}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m} (\alpha+\beta)_{2m+n} (\beta)_n}{(1-\alpha)_{2m} (\delta)_m (1+\beta-\gamma)_n} (-1)^n \frac{(x)^m}{m!} \frac{(1/y)^n}{n!}$$

$$- \frac{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma)}{\Gamma(1-\alpha-\gamma)\,\Gamma(\beta)} (y)^{-\gamma}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m} (\alpha+\gamma)_{2m+n} (\gamma)_n}{(1-\alpha)_{2m} (\delta)_m (1+\gamma-\beta)_n} (-1)^n \frac{(x)^m}{m!} \frac{(1/y)^n}{n!} \tag{72}$$

3. CONCLUSION

We have obtained asymptotic expansions for a class of hypergeometric functions where the magnitude of one of the variables is large while that of the other is small. Using the Barnes-type integral representation the hypergeometric function is analytically continued to the region of interest. The poles that occur are assumed to be simple. The case when the poles are not simple will be considered in another report.

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